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Approximation of dual Gabor frames, window decay, and wireless communications

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Abstract

We consider three problems for Gabor frames that have recently received much attention. The first problem concerns the approximation of dual Gabor frames in $L_2(\mathbb{R})$ by finite-dimensional methods. Utilizing Wexler-Raz type duality relations we derive a method to approximate the dual Gabor frame, that is much simpler than previously proposed techniques. Furthermore it enables us to give estimates for the approximation rate when the dimension of the finite model approaches infinity. The second problem concerns the relation between the decay of the window function g and its dual γ. Based on results on commutative Banach algebras and Laurent operators we derive a general condition under which the dual γ inherits the decay properties of g. The third problem concerns the design of pulse shapes for orthogonal frequency division multiplex (OFDM) systems for time- and frequency dispersive channels. In particular, we provide a theoretical foundation for a recently proposed algorithm to construct orthogonal transmission functions that are well localized in the time-frequency plane.

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1   Introduction

Gabor systems play an important role in signal processing and digital communication. In filter bank theory they are known under the name *oversampled modulated filter banks* [6], in wireline communications they correspond to the concept of *discrete multitone transmultiplexing*, and in wireless communications they are (implicitly) used in *orthogonal frequency division multiple access systems* [34, 25, 1].

A Gabor system consists of functions of the form

$$g_{na,mb}(t) = e^{2\pi imb}g(t - na), \ \ n, m \in \mathbb{Z}, \ a, b \in \mathbb{R}, \tag{1}$$

where $g \in \boldsymbol{L}_2(\mathbb{R})$ is – depending on the context – called *window*, *atom*, or *pulse shape*. The parameters a and b represent the time-shift and frequency-shift, respectively.

We say that (g, a, b) generates a *Gabor frame* for $\boldsymbol{L}_2(\mathbb{R})$ for given shift parameters a, b if there exist constants (*frame bounds*) $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{n,m\in\mathbb{Z}} |\langle f, g_{na,mb}\rangle|^2 \le B\|f\|^2, \tag{2}$$

for any $f \in \boldsymbol{L}_2(\mathbb{R})$.

The analysis operator T is defined as

$$T : f \in \boldsymbol{L}_2(\mathbb{R}) \rightarrow Tf = \{\langle f, g_{na,mb}\rangle\}_{n,m\in\mathbb{Z}}, \tag{3}$$

and the synthesis operator, which happens to be the adjoint of T is

$$T^* : c \in \boldsymbol{\ell}_2(\mathbb{Z} \times \mathbb{Z}) \rightarrow T^*c = \sum_{n,m\in\mathbb{Z}} c_{nm} g_{na,mb}. \tag{4}$$

The Gabor frame operator is defined by

$$Sf = \sum_{n,m\in\mathbb{Z}} \langle f, g_{na,mb}\rangle g_{na,mb}, \qquad f \in \boldsymbol{L}_2(\mathbb{R}), \tag{5}$$

and satisfies

$$IA \le S \le IB$$

where I is the identity operator on $\boldsymbol{L}_2(\mathbb{R})$. Of course $S = T^*T$.

If (g, a, b) establishes a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ then any f in $\boldsymbol{L}_2(\mathbb{R})$ can be represented as

$$f = \sum_{n,m \in \mathbb{Z}} \langle f, \gamma_{n,m} \rangle g_{na,mb} = \sum_{n,m \in \mathbb{Z}} \langle f, g_{na,mb} \rangle \gamma_{n,m}, \qquad (6)$$

where the *dual frame* $\{\gamma_{n,m}\}$ is given by

$$\gamma_{n,m} = e^{2\pi imb} \gamma(t - na), \;\; n, m \in \mathbb{Z}; \; a, b \in \mathbb{R},$$

with $\gamma = S^{-1}g$. In general there are many functions generating dual frames that satisfy relation (6). The "canonical" dual window γ has several nice properties. One of them is that it has minimal $\boldsymbol{L}_2$-norm among all dual functions. In this paper we concentrate on the canonical dual window and henceforth simply talk about *the* dual window and *the* dual frame. For more details about the properties of Gabor frames and their duals the reader is referred to [8, 14].

The rest of the paper is organized as follows. In Section 2 we analyze the problem of approximating the dual window by using finite-dimensional methods. Based on Wexler-Raz type duality relations we derive a method to approximate the dual Gabor frame, that is much simpler than previously proposed techniques. Furthermore we show that for windows with exponential decay in time and frequency the proposed approach yields an exponential approximation rate when the dimension of the finite model approaches infinity. In Section 3 we dig deeper into decay properties of g and its dual γ. Based on results on commutative Banach algebras and Laurent operators we derive a general condition on the decay of g which guarantees that the dual γ inherits these decay properties. Finally in Section 4 we demonstrate the relevance of the results derived in Section 3 for wireless communications. In particular, we provide a theoretical foundation for a recently proposed algorithm to construct an orthogonal frequency division multiplex (OFDM) system with good time-frequency localization properties.

Before we proceed we introduce a few notations used throughout the paper.

The Fourier transform of a function f is given by

$$\hat{f}(\omega) = \int\limits_{-\infty}^{+\infty} f(t) e^{-2\pi it\omega} dt.$$

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The *short time Fourier transform* (STFT) of f with respect to the (sufficiently nice) window g is

$$(\mathcal{V}_g f)(t, \omega) = \int\limits_{-\infty}^{+\infty} f(x)\overline{g(x - t)}e^{-2\pi i x \omega}\, dx. \tag{7}$$

A locally integrable function w is called *weight function*, if w is positive and submultiplicative, i.e., if $w(t) > 0$ and $w(t_1 + t_2) \leq w(t_1)w(t_2)$.

The space $\boldsymbol{L}_{1,w}(\mathbb{R})$ consists of all functions f with

$$\int\limits_{-\infty}^{+\infty} |f(t)|w(t)dt < \infty,$$

where w is a weight function. Similarly $\boldsymbol{\ell}_{1,w}(\mathbb{Z})$ consists of all sequences $x = \{x_k\}_{k \in \mathbb{Z}}$ with

$$\sum_{k=\infty}^{\infty} |x_k|w(k) < \infty.$$

It is convenient to define following spaces:

$$\mathcal{C}_{[\alpha,\beta]} = \left\{ f \in \boldsymbol{L}_2(\mathbb{R}) : \exists\, \alpha, \beta \text{ such that } |f(t)| = 0 \ , \forall |t| \notin [\alpha, \beta] \right\}.$$

$$\mathcal{E}_\lambda = \left\{ f \in \boldsymbol{L}_2(\mathbb{R}) : \exists\, \lambda > 0, c > 0 \text{ such that } |f(t)| \leq ce^{-\lambda|t|} \ , \forall |t| \right\}.$$

$$\mathcal{Q}_s = \left\{ f \in \boldsymbol{L}_2(\mathbb{R}) : \exists\, s > 1, c > 0 \text{ such that } |f(t)| \leq c(1 + |t|)^{-s} \ , \forall |t| \right\}.$$

Finally, the Moore-Penrose inverse [12] of a bounded operator T is denoted by T^+.

2 Gabor frames, finite sections and the duality condition

The theoretical concepts for Gabor analysis are usually developed for infinite-dimensional function spaces, notably for $\boldsymbol{L}_2(\mathbb{R})$ or $\boldsymbol{\ell}_2(\mathbb{Z})$, whereas all numerical implementations have to be done within a finite-dimensional framework. The connection between Gabor systems on $\boldsymbol{\ell}_2(\mathbb{Z})$ and several finite-dimensional models has been clarified in [29, 31]. In this section we extend

these results to Gabor frames for $\boldsymbol{L}_2(\mathbb{R})$. Different techniques than those used for $\boldsymbol{\ell}_2(\mathbb{Z})$ are required for $\boldsymbol{L}_2(\mathbb{R})$.

Our approach relies on a remarkable property of Gabor frames, whose discovery has its origin in a paper by Wexler and Raz [32], their result was later made precise and extended by Janssen [24], Ron and Shen [28], and Daubechies, H. Landau, and Z. Landau [11].

For given g, a, b we define the operator H by

$$Hf := \{\langle f, g_{k/b,l/a}\rangle\}_{k,l\in\mathbb{Z}}, \qquad f \in \boldsymbol{L}_2(\mathbb{R}),$$

where

$$g_{k/b,l/a}(t) = g(t - k/b)e^{2\pi itl/a}.$$

The adjoint H^* of H is

$$H^*c = \sum_{k,l} c_{k,l} g_{k/b,l/a}, \qquad c = \{c_{k,l}\}_{k,l\in\mathbb{Z}} \in \boldsymbol{\ell}^2(\mathbb{Z}^2).$$

We identify HH^* with its matrix representation, given by

$$HH^* = \{\langle g_{k'/b,l'/a}, g_{k/b,l/a}\rangle\}_{k,l;k',l'\in\mathbb{Z}}.$$

The Wexler-Raz biorthogonality relations imply that the dual window satisfies [24, 11]:

$$\gamma = H^*(HH^*)^{-1}\sigma = ab \sum_{k,l\in\mathbb{Z}} [(HH^*)^{-1}]_{k,l;0,0} \ g_{k/b,l/a}, \qquad (8)$$

where $\sigma = \{(ab)\delta_{k0}\delta_{l0}\}_{k,l\in\mathbb{Z}}$.

Furthermore, there holds:

Theorem 2.1 ([28]) $(g, 1/b, 1/a)$ *is a Riesz basis for its closed linear span if and only if* (g, a, b) *is a frame for* $\boldsymbol{L}_2(\mathbb{R})$.

Formula (8) and Theorem 2.1 are the main ingredients for our approach to approximate the dual window γ using a finite-dimensional model.

Now, for $x \in \boldsymbol{\ell}^2(\mathbb{Z}^2)$ and $n \in \mathbb{N}$ define the orthogonal projections P_n by

$$(P_n x)_{k,l} = \begin{cases} x_{k,l} & \text{if } \max\{|k|, |l|\} \leq n, \\ 0 & \text{else.} \end{cases}$$

We identify the image of P_n with the $(2n+1)^2$-dimensional space $\mathbb{C}^{(2n+1)^2}$ and write

$$H_n f := P_n H f = \{\langle f, g_{k/b,l/a}\rangle\}_{|k|,|l|\leq n}.$$

The matrix

$$H_n H_n^* = (HH^*)_n = P_n HH^* P_n = \{\langle g_{k'/b,l'/a}, g_{k/b,l/a}\rangle\}_{|k|,|l|;|k'|,|l'|\leq n}.$$

is a finite section of the infinite-dimensional Gram matrix HH^*.

We say that the finite section method is *applicable* to HH^*, if, beginning with some $n \in \mathbb{N}$, for each $y \in \text{range}(H)$ the equation

$$H_n^*[H_n H_n^*]^{-1} x^{(n)} = P_n y$$

has a unique solution $x^{(n)} \in \text{Im}(P_n)$ and as $n \to \infty$ the vectors $x^{(n)}$ tend to the solution of $HH^* x = y$.

We set

$$\gamma^{(n)} := H_n^*[H_n H_n^*]^{-1} \sigma^{(n)}$$

for $n = 0, 1, 2, \ldots$.

There holds:

Theorem 2.2 *Let (g, a, b) generate a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ and let γ be the dual window. Then*

$$\|\gamma - \gamma^{(n)}\| \to 0 \qquad \text{for } n \to \infty.$$

We need following result for the proof of Theorem 2.2.

Lemma 2.3 (Lemma of Kantorovich, [27]) *Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of invertible operators on a Banach space $\boldsymbol{B}$, and assume that the sequence is uniformly bounded above and below, i.e, that there exist constants $0 < C_1, C_2 < \infty$ such that*

$$C_1 \|f\|_{\boldsymbol{B}} \leq \|K_n f\|_{\boldsymbol{B}} \leq C_2 \|f\|_{\boldsymbol{B}} \qquad \text{for all } n\in\mathbb{N} \text{ and } f \in \boldsymbol{B}. \qquad (9)$$

Then $\{K_n\}$ converges in the strong operator topology to an invertible operator if and only if the same is true for $\{K_n^{-1}\}_n$, and then

$$\{\lim K_n\}^{-1} = \lim\{K_n^{-1}\}.$$

Proof of Theorem 2.2:

Theorem 2.1 implies that $(g, 1/b, 1/a)$ is a Riesz basis for its closed span with *Riesz bounds* A, B. Any finite subset of a Riesz basis is again a Riesz basis for its closed span, cf. Chapter 1 in [33]. Thus for $\{g_{k/b,l/a}\}_{|k|,|l|\leq n}, n \in \mathbb{N}$ there exist constants A_n, B_n such that

$$A \leq A_n \leq H_n H_n^* \leq B_n \leq B. \tag{10}$$

Hence we can apply Lemma 2.3 to establish that the finite section method is applicable to HH^*.

Finally note that H_n obviously converges to H pointwise for $n \to \infty$, thus $\|\gamma^{(n)} - \gamma\| \to 0$ for $n \to \infty$. $\qquad\blacksquare$

It is interesting to ask if we can give some estimate on the rate of approximation of the finite section method in case the window g satisfies certain decay conditions in time and/or frequency. In the following theorem we concentrate on windows with exponential decay in time and frequency domain, but it will become clear from the proof that similar results may be obtained for other types of decay.

Theorem 2.4 *Let (g, a, b) be a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ and assume that there exist constants $C, D > 0$ such that*

$$|g(t)| \leq Ce^{-\lambda|t|}, \quad and \quad |\hat{g}(\omega)| \leq De^{-\lambda|\omega|}, \tag{11}$$

then there exists a $\lambda' < \lambda$ and a constant C' depending on the frame bounds and on λ', but independent of n such that

$$\|\gamma - \gamma^{(n)}\| \leq C'e^{-\lambda'n}.$$

Proof: First we show that (11) implies that the entries of HH^* satisfy

$$(HH^*)_{kl;k'l'} \leq D_2 e^{-\lambda_1(|k-k'|+|l-l'|)} \tag{12}$$

for some $\lambda_1 < \lambda$ and some constant D_2.

It is clear that

$$|(HH^*)_{kl,k'l'}| = |\langle g_{k/b,l/a}, g_{k'/b,l'/a}\rangle| = |\langle g, g_{k-k',l-l'}\rangle| \tag{13}$$

and

$$\langle g, g_{k/b,l/a}\rangle = (\mathcal{V}_g g)(k/b, l/a) \quad k, l \in \mathbb{Z}. \tag{14}$$

Now

$$|(\mathcal{V}_g g)(t, \omega)| = \left| \int\limits_{-\infty}^{+\infty} g(x)\overline{g(x-t)}e^{-2\pi i \omega x} \right| dx \leq \int\limits_{-\infty}^{+\infty} |g(x)||\overline{g(x-t)}|dx$$

$$\leq C^2 \int\limits_{-\infty}^{+\infty} e^{-\lambda|x|}e^{-\lambda|x-t|}dx \leq C^2 \int\limits_{-\infty}^{+\infty} e^{-\lambda|x|}e^{-\lambda_1|x-t|}dx \tag{15}$$

for some $\lambda_1 < \lambda$. Set $\varepsilon = \lambda - \lambda_1$, then

$$C^2 \int\limits_{-\infty}^{+\infty} e^{-\lambda|x|}e^{-\lambda_1|x-t|}dx \leq C^2 \int\limits_{-\infty}^{+\infty} e^{-\lambda|x|}e^{-\lambda_1(|t|-|x|)}dx$$

$$\leq C^2 \int\limits_{-\infty}^{+\infty} e^{-\varepsilon|x|}e^{-\lambda_1|t|}dx \leq\leq C^2 e^{-\lambda_1|t|} \int\limits_{-\infty}^{+\infty} e^{-\varepsilon|x|}dx$$

$$\Rightarrow |(\mathcal{V}_g g)(t, \omega)| \leq C_1 e^{-\lambda_1|t|} \tag{16}$$

for some constant C_1 depending on λ_1.

Similarly

$$|(\mathcal{V}_g g)(t, \omega)| = |(\mathcal{V}_{\hat{g}}\hat{g})(\omega, -t)| \leq D_1 e^{-\lambda_1|\omega|}. \tag{17}$$

By combining (16) and (17) and taking the square root we get

$$|(\mathcal{V}_g g)(t, \omega)| \leq \sqrt{C_1 D_1}e^{-\frac{\lambda_1}{2}(|t|+|\omega|)}. \tag{18}$$

This together with equations (13) and (14) yields (12).

Now consider

$$\|\gamma - \gamma^{(n)}\| \leq \|H^*(HH^*)^{-1}\sigma - H^*(HH^*)^{-1}H_n H_n^*(H_n H_n^*)^{-1}\sigma^{(n)}\| +$$
$$\|H^*(HH^*)^{-1}H_n H_n^*(H_n H_n^*)^{-1}\sigma^{(n)} - H^*(HH^*)^{-1}HH_n^*(H_n H_n^*)^{-1}\sigma^{(n)}\|$$
$$\leq \underbrace{\|H^*(HH^*)^{-1}(\sigma - \sigma^{(n)})\|}_{0} + A^{-1}\|(H_n H_n^* - HH_n^*)(H_n H_n^*)^{-1}\sigma^{(n)}\|$$
$$= A^{-1}\|(H_n H_n^* - HH_n^*)(H_n H_n^*)^{-1}\sigma^{(n)}\|.$$

Note that $(H_n H_n^* - HH_n^*)$ is a matrix that has a finite number of columns and a biinfinite number of rows, of which the rows with index $|kl| \leq n$ are zero.

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By (12) the entries of $H_n H_n^*$ and the nonzero rows of $(H_n H_n^* - H H_n^*)$ decay exponentially off the diagonal. Relation (10) implies

$$\text{cond}(H_n H_n^*) \leq \text{cond}(H H^*) \quad \text{for } n = 0, 1, \ldots. \tag{19}$$

Using (19) and Proposition 2 in [23] it follows that there exists a λ_2 and a constant C_2 depending on λ_2 and on the condition number of $H H^*$, such that

$$|[(H_n H_n^*)^{-1}]_{k,l;0,0}| \leq C_2 e^{-\lambda_2(|k|+|l|)},$$

and the same is true for $v^{(n)} := (H_n H_n^* - H H_n^*)[(H_n H_n^*)^{-1}]_{k,l;0,0}$. Now it is easy to see that there exist λ' and C' such that

$$\|v^{(n)}\| = \sum_{|k|,|l|>n} |v_{kl}^{(n)}|^2 \leq C' e^{-\lambda' n}.$$

$$\blacksquare$$

For more results on the connection between the decay of a function and its short time Fourier transform the reader is referred to [20].

To fully appreciate the simplicity and advantages of the proposed approach we consider for comparison the following method for determining the "dual" expansion coefficients $\langle f, \gamma_{n,m} \rangle$.

The function $f \in \boldsymbol{L}_2(\mathbb{R})$ can be expressed as

$$f = \sum_{k,l \in \mathbb{Z}} c_{n,m} g_{na,mb} \tag{20}$$

$$\text{where } (TT^*)^+ c = d, \tag{21}$$

with $d = \{d_{n,m}\} = \{\langle f, g_{na,mb} \rangle\} = Tf$. We can identify TT^* with its Gram matrix representation $(TT^*)_{m,n} = \langle g_{n'a,m'b}, g_{na,mb} \rangle$ for $m, n, m', n' \in \mathbb{Z}$.

Setting $T_n T_n^* = P_n TT^* P_n$ and $d^{(n)} = P_n d$, we obtain the n-th approximation $c^{(n)}$ to c by solving

$$T_n T_n^* c^{(n)} = d^{(n)}. \tag{22}$$

Unfortunately the (generalized) inverse of $T_n T_n^*$ is not bounded for $n \to \infty$, although $\|(TT^*)^+\| \leq A^{-1}$. In fact, $\|(T_n T_n^*)^+\| \to \infty$ for $n \to \infty$. Hence $c^{(n)}$ does not converge to c and without further modifications this approach does not lead to an approximation of f.

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Instead of computing $(T_n T_n^*)^+$ we can compute a regularized inverse via a truncated singular value decomposition by setting the singular values of TT^* below a certain threshold τ to zero. Let $(T_n T_n^*)^{\tau,+}$ denote this regularized inverse. It is shown in [22] that $(T_n T_n^*)^{\tau,+}$ converges strongly to $(TT^*)^+$ if we allow the threshold parameter to vary for each n.

In order to use this approach for practical purposes we need good estimates for the optimal threshold τ_n. Assuming a numerical precision of δ of the data and setting $B_n = \|T_n T_n^*\|$, it is shown in [30] that τ_n can be estimated by

$$\tilde{\tau}_n \leq B_n \left(\frac{\delta}{p}\right)^{\frac{1}{p+1}} .$$

Here – without going into details about regularization theory – p can be seen as "smoothness parameter" [12], the standard setting for p in regularization theory is $p = 1$ or $p = 2$. Thus for large n we get

$$\tilde{\tau} \in [B(\delta/2)^{\frac{1}{3}}, B\delta^{\frac{1}{2}}]$$

where B is the upper frame bound. Good estimates for the upper frame bound are important to apply this method in practice.

All this extra effort is not necessary when using Theorem 2.2. A different approach has been proposed in [4], which however requires substantially more effort compared to the Wexler-Raz based approach in this section.

3 Laurent operators and decay of dual Gabor frames

A natural question for Gabor frames – also in spite of Theorem 3.4 is the following. Given a window g with certain decay properties in time and/or frequency, does its dual γ have the same decay properties? This question is not only of interest from a theoretical viewpoint, but has a number of practical implications, e.g., see Section 4.

We briefly summarize a few important results. We assume in the sequel that $\{g_{na,mb}\}$ constitutes a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ or $\boldsymbol{\ell}_2(\mathbb{Z})$.
(i) If $g \in \mathcal{C}_{[\alpha,\beta]}$, then $\gamma \in \mathcal{C}_{[\alpha_1,\beta_1]}$ only in very special cases [2]. In general γ is no longer compactly supported, but has exponential decay (see [29] for a proof in $\boldsymbol{\ell}_2(\mathbb{Z})$, the result can be easily extended to $\boldsymbol{L}_2(\mathbb{R})$). Hence in this case g and γ do not belong to the same type of space.

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(ii) If $g \in \mathcal{E}_\lambda$ then $\gamma \in \mathcal{E}_{\lambda_1}$, but in general with a different exponent $\gamma_1 < \gamma$ [29, 3]. Thus g and γ have the same type of decay, but do not belong to the same space, we lose some quality of decay.[1]

(iii) If $g \in \mathcal{Q}_s$, then $\gamma \in \mathcal{Q}_s$ (see [29] for a proof for $\ell_2(\mathbb{Z})$, the result can be easily generalized to $\boldsymbol{L}_2(\mathbb{R})$ using the same approach as in [3]). In this case g and γ actually belong to the same space $\mathcal{Q}_s$.

From an algebraic point of view case (iii) is the most appealing one. Can we find a (simple) condition on the decay of g that implies that the dual γ belongs to the same space, similar to case (iii)? In this section we will give an exhaustive answer to this question.

We need some preparation before we proceed.

Definition 3.1 *Let $w(t)$ be a continuous weight function on $\mathbb{R}$. The (weighted) Wiener Algebra $\mathcal{A}_w$ is the Banach space of absolutely convergent Fourier series of period 1 (cf. [26]), i.e., $f \in \mathcal{A}_w$ if*

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t} \tag{23}$$

with

$$\sum_{k=-\infty}^{\infty} |c_k| w(k) < \infty. \tag{24}$$

The norm on $\mathcal{A}_w$ is

$$\|f\|_{\mathcal{A}_w} = \sum_{k=-\infty}^{\infty} |c_k| w_k.$$

It follows from Chapter 19.4 of [17] that $\mathcal{A}_w$ is a Banach algebra under pointwise multiplication. $\mathcal{A}_w$ can be identified with the space of all sequences $c = \{c_k\}$ which are in $\ell_{1,w}$.

Definition 3.1 can be extended to matrix-valued functions in a straightforward manner. Put

$$\Phi(e^{2\pi i t}) = \begin{bmatrix} f_{11}(e^{2\pi i t}) & \cdots & f_{1m}(e^{2\pi i t}) \\ \vdots & & \vdots \\ f_{m1}(e^{2\pi i t}) & \cdots & f_{mm}(e^{2\pi i t}) \end{bmatrix} \tag{25}$$

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[1]Note that there do exist Gabor frames with Gaussian decay, that have *non-canonical* duals with Gaussian decay, see Example 3.10 in [3].

and set

$$A_k = \int_0^1 \Phi(e^{2\pi it})e^{-2\pi it}dt. \qquad (26)$$

Let the weight function $w(t)$ act on $\mathbb{R}^m$. Then Φ belongs to the matrix Wiener algebra $\mathcal{A}_w^m$ if

$$\sum_{k=-\infty}^{\infty} |A_k|w(k) < \infty. \qquad (27)$$

Φ is unitarily equivalent to the block Laurent operator L whose matrix representation (with respect to the standard basis) is given by

$$\begin{bmatrix} \ddots & & & & \\ & A_0 & A_{-1} & A_{-2} & \\ & A_1 & \boxed{A_0} & A_{-1} & \\ & A_2 & A_1 & A_0 & \\ & & & & \ddots \end{bmatrix}.$$

Here $\boxed{A_0}$ denotes the $(0,0)$ entry which acts on the 0-th coordinate space. Φ is also called the defining function of the block Laurent operator L. By a slight abuse of notation we also write $L = [A_{kl}]_{k,l=-\infty}^{\infty}$ where $A_{kl} = A_{k-l}$.

We define $\mathcal{W}_w^m$ as the space consisting of all block Laurent operators whose $m \times m$ blocks A_k satisfy (27). If $m = 1$ we simply write $\mathcal{W}_w$ and L reduces to a scalar-valued Laurent operator.

Theorem 3.2 *Let $L \in \mathcal{W}_w$ be self-adjoint and positive definite. Assume that the weight function w satisfies*

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{w(-n)}} = 1 \quad and \quad \lim_{n\to\infty} \sqrt[n]{w(n)} = 1, \qquad (28)$$

then $L^{-1} \in \mathcal{W}_w$.

Proof: Set $L = [A_{kl}]_{k,l=-\infty}^{\infty}$ and $L^{-1} = [B_{kl}]_{k,l=-\infty}^{\infty}$. Since L is positive definite we have

$$f(\omega) = \sum_{k=-\infty}^{\infty} A_k e^{2\pi ik\omega} > 0 \qquad (29)$$

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and by the properties of Laurent operators [18]

$$\frac{1}{f(\omega)} = \sum_{k=-\infty}^{\infty} B_k e^{2\pi i k\omega}.$$

The property $L \in \mathcal{W}_w$ is equivalent to $f \in \mathcal{A}_w$. By Theorem 2 on page 24 in [17] an element of $\mathcal{A}_w$ has an inverse in $\mathcal{A}_w$ if it is not contained in a maximal ideal. Any maximal ideal of $\mathcal{A}_w$ consists of elements of the form (cf. Chapter 19.4 in [17])

$$\sum_{k=-\infty}^{\infty} a_k \xi^k = 0,$$

where $\xi = \rho e^{2\pi i\omega}$ with

$$\rho_1 \le \rho \le \rho_2$$

and

$$\rho_1 = \lim_{n\to\infty} \frac{1}{\sqrt[n]{w_{-n}}} \quad \text{and} \quad \rho_2 = \lim_{n\to\infty} \sqrt[n]{w_n}.$$

Due to assumption (28) we get $\rho_1 = \rho_2 = 1$, hence $\rho = 1$. Thus a necessary and sufficient condition for an element in $\mathcal{A}_w$ to be not contained in a maximal ideal is $\sum_k a_k e^{2\pi i k\omega} \neq 0$ for all ω. By assumption L is positive definite, hence $f(\omega) = \sum_k A_k e^{2\pi i k\omega} > 0$ for all ω, consequently $L^{-1} \in \mathcal{W}_w$. ❏

The proof of Theorem 3.2 is essentially based on results of Gelfand, Raikov, and Shilov [17]. A crucial role plays condition (28), to which we will henceforth refer as the *Gelfand-Raikov-Shilov condition* (GRS-condition for short) .

Now let L be a block Laurent operator with defining function $\Phi = [f_{ij}]_{i,j=1}^m \in \mathcal{A}_w^m$. If L is hermitian, positive definite then

$$\det([f_{ij}(\lambda)]_{i,j=1}^m) > 0, \qquad |\lambda| = 1.$$

Hence we can apply Theorem 8.1 on page 830 in [18] to extend Theorem 3.2 to block Laurent operators and obtain the following

Corollary 3.3 *Let $L \in \mathcal{W}_w^m$ be self-adjoint and positive definite. Assume that the weight function w defined on $\mathbb{R}^m$ satisfies the GRS-condition, i.e.,*

$$\lim_{n \to \infty} \frac{1}{\sqrt[n]{w(-n)}} = 1 \quad and \quad \lim_{n \to \infty} \sqrt[n]{w(n)} = 1, \tag{30}$$

then $L^{-1} \in \mathcal{W}_w^m$.

It is clear that in the derivations above we can replace 1-periodic functions by α-periodic functions ($\alpha \in \mathbb{Q}$). Similarly, sequences with indices in $\mathbb{Z}$ can be replaced by sequences indexed by $\mathbb{Z}_\alpha$ (i.e. by indices of the form $k\alpha$).

Now we are ready to prove a general result on the decay properties of Gabor windows g and their duals γ.

Theorem 3.4 *Let (g, a, b) generate a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ with $ab = \frac{p}{q} \in \mathbb{Q}$ with relative prime integers p and q. Let $g \in \boldsymbol{L}_{1,w}(\mathbb{R})$ where w satisfies the GRS-condition. Then $\gamma \in \boldsymbol{L}_{1,w}(\mathbb{R})$.*

Proof: The assumption $g \in \boldsymbol{L}_{1,w}(\mathbb{R})$ implies $\sum_{k=-\infty}^{\infty} |g(k)|w(k) < \infty$ and $\sum_{k=-\infty}^{\infty} |g(k/b)|w(k/b) < \infty$ for $b \in \mathbb{N}$. Denote

$$G_{kl}(t) = \frac{1}{b} \sum_{r=-\infty}^{\infty} g(t - ra - \frac{k}{b})g^*(t - ra - \frac{l}{b}), \qquad k, l \in \mathbb{Z}. \tag{31}$$

$G(t)$ is periodic in t with period a. Since $ab = \frac{p}{q}$ we also get $G_{kl}(t) = G_{k+p,l+p}(t)$, cf. [13], Theorem 3.2. In words, $G(t)$ is a block Laurent operator. The frame property implies that $G(t)$ is hermitian positive definite and

$$AI \leq G(t) \leq BI.$$

The submultiplicativity of the function w implies that the spaces $\boldsymbol{L}_{1,w}(\mathbb{R})$, $\boldsymbol{\ell}_{1,w}(\mathbb{Z})$, and $\boldsymbol{\ell}_{1,w}(\mathbb{Z}_{\frac{1}{b}})$ are Banach algebras under convolution (cf. [17]). Hence $\{G_{0k}(t)\}_{k \in \mathbb{Z}} \in \boldsymbol{\ell}_{1,w}(\mathbb{Z}_{\frac{1}{b}}^m)$, and $G(t) \in \mathcal{W}_w^m$. By Corollary 3.3

$$G^{-1}(t) \in \mathcal{W}_w^m. \tag{32}$$

The dual γ can be expressed as (see e.g. [3])

$$\gamma(t) = \sum_{k=-\infty}^{\infty} [G^{-1}(t)]_{0k} g(t - \frac{k}{b}). \tag{33}$$

This equation together with (32) and the assumption $g \in \boldsymbol{L}_{1,w}(\mathbb{R})$ yields that $\gamma \in \boldsymbol{L}_{1,w}(\mathbb{R})$. $\qquad \qquad \blacksquare$

Remark: (i) The idea to exploit the Laurent operator property of the Gabor frame operator in the context of window decay is not new. It has been used by Feichtinger and Gröchenig [13] in connection with modulation spaces and polynomial weights.
(ii) It is easy to reformulate Theorem 3.4 for windows whose decay properties are given in the frequency domain. We leave this modification to the reader.

Observe that condition (28) is satisfied e.g. for $w(x) = (1 + |x|)^s$ and $w(x) = e^{\lambda |x|^\gamma}, \gamma < 1$, but not for $w(x) = e^{\lambda |x|}$. This is why we have to use a smaller exponent $\lambda_1 < \lambda$ for the dual to bound the decay in case of an exponentially decaying window.

A detailed discussion on the connection between decay properties of windows and modulation spaces can be found in [19].

4 Orthogonal frequency division multiplexing and Gabor systems

Orthogonal frequency division multiplexing (OFDM) has attracted a great deal of attention as an efficient technology for wireless data transmission [34]. Among others it is currently used in the European digital audio broadcasting standard [5]. In a wireline environment OFDM is known under the name *discrete multitone transmultiplexing* (DMT).

The basic idea of OFDM is to divide the available spectrum into several subchannels (subcarriers). A baseband OFDM system is schematically represented in Figure 1.
Transmitter: Assuming N subcarriers, a bandwidth of W Hz, *symbol length* of T seconds, and *carrier separation* $F := W/N$, the transmitter of a general OFDM system uses the following waveforms

$$\psi_l(t) = \psi(t)e^{2\pi itlF}, \quad k \in \mathbb{Z}, l = 0, \dots, N-1. \tag{34}$$

The transmitted baseband signal for OFDM symbol number k is

$$s_k(t) = \sum_{l=0}^{N-1} c_{kl}\psi_l(t - kT), \tag{35}$$

where $c_{k0}, c_{k1}, \dots, c_{k,N-1}$ are the complex-valued information bearing coefficients (data symbols). Assuming an infinite sequence of OFDM symbols is
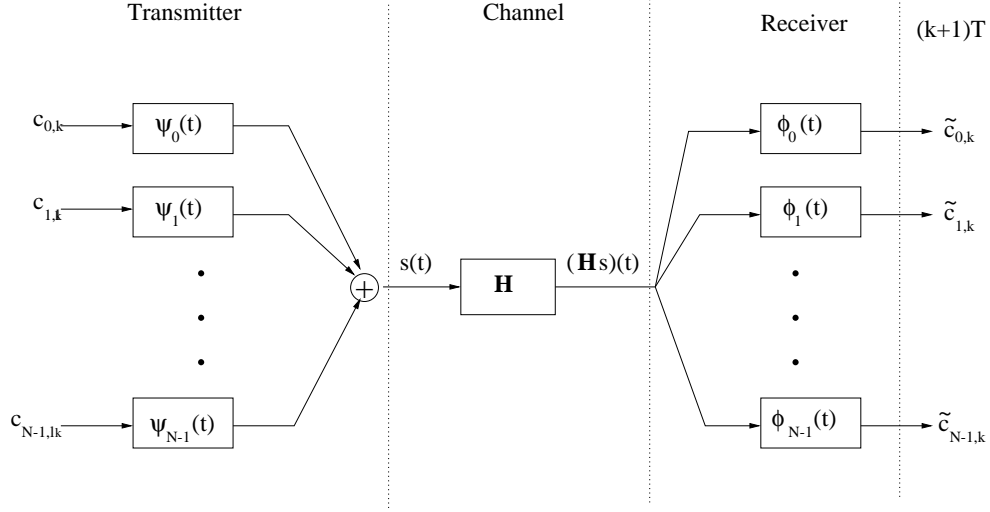
Figure 1: A general baseband OFDM system configuration

transmitted, the output from the transmitter is a superposition of individual OFDM symbols

$$s(t) = \sum_{k=-\infty}^{\infty} s_k(t) = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} c_{k,l} \psi_l(t - kT). \tag{36}$$

Receiver: The OFDM receiver consists of a matched filter bank $\{\phi_l\}$ of similar structure as the transmitter waveforms, i.e.,

$$\phi_{kl}(t) = \phi(t - kT)e^{2\pi itlF}, \quad k \in \mathbb{Z}, l = 0, \ldots, N-1. \tag{37}$$

The transmitted data are recovered by projecting the received signal $r = \mathbf{H}s + \nu$ (where ν represents additive white Gaussian noise, AWGN for short) onto the functions ϕ_{kl}, i.e.,

$$\tilde{c}_{kl} = \langle r, \phi_{kl} \rangle. \tag{38}$$

In the standard OFDM setup the functions ψ_{kl} are designed to be mutually orthogonal. In this case $\phi \equiv \psi$. The situation where the sets $\{\psi_{kl}\}$ and $\{\phi_{kl}\}$ are biorthogonal is referred to as *biorthogonal frequency division multiplexing* (BFDM).

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It is useful to have the following two OFDM setups at hand.

Continuous-time model: Since $s(t)$ is a continuous signal, it is convenient to consider a continuous-time transmission set that is infinite in both indices k and l, which means we set $c_{kl} = 0$ if $l < 0$ or $l > N - 1$. Using the viewpoint it is clear that OFDM can be interpreted as an orthonormal Gabor system in $\boldsymbol{L}_2(\mathbb{R})$. For a theoretical analysis it is often advantageous to work with this time-continuous model.

Discrete-time model: In practice the OFDM system is digitally implemented. At the transmitter the signal s is passed through a digital-to-analog converter to obtain a continuous signal. At the receiver an analog-to-digital converter transforms the received signal back into a discrete-time signal. Therefore it is useful to consider the following discrete-time setup for OFDM.

Let $\psi \in \boldsymbol{\ell}_2(\mathbb{Z})$ and set $\psi_{k,l}(n) = \psi(n - kT)e^{2\pi i l F}$ for $k, n \in \mathbb{Z}, l = 0, \ldots, N - 1$ and $F = W/N$. Then the OFDM signal becomes

$$s(n) = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} c_{k,l} \psi_{k,l}. \tag{39}$$

It is clear that $\{\psi_{k,l}\}$ coincides with a discrete-time Gabor system.

4.1 Wireless channels and time-frequency localization

In the ideal case the channel $\mathbf{H}$ does not introduce any distortion, and the data c_{kl} can be recovered exactly. However in practice wireless channels introduce time dispersion as well as frequency dispersion (in addition to the usual channel noise). The time dispersion is caused by multipath propagation and can lead to intersymbol interference (ISI). Frequency dispersion of the mobile radio channel is due to the Doppler effect and can cause interchannel interference (ICI). The distortion resulting from channel dispersion depends crucially on the time-frequency localization of the transmitter pulse shapes $\{\phi_{kl}\}$. Robustness against doubly dispersive channels can be achieved by pulse shapes with good time-frequency localization.

An optimum OFDM system in case of doubly dispersive channels would consist of orthogonal basis functions with $TF = 1$, such that the $\psi_{k,l}$ are well localized in time and frequency. The condition $TF = 1$ (critical sampling) ensures maximal spectral efficiency of the transmission system. Unfortunately, such a system cannot exist due to the Balian-Low theorem [9].

Therefore other approaches (cyclic prefix, pulse-shaping, BFDM) have been proposed for doubly dispersive channels [34, 21, 25, 1]. These approaches can be interpreted in the time-frequency plane as using an undersampled grid, i.e., $TF > 1$, which results in a set of basis functions that is incomplete in $\boldsymbol{L}_2(\mathbb{R})$ (or $\boldsymbol{\ell}_2(\mathbb{Z})$, respectively). Although the choice $TF > 1$ leads to a loss in the capacity of the transmission system (determined by the undersampling rate), it is usually an acceptable price to pay to mitigate interference.

We know from Gabor theory that there exist Gabor systems with good time-frequency localization for $TF > 1$ (take for instance $TF = 2$ and a Gaussian window). In the present context of OFDM we are particularly interested in the construction of orthonormal incomplete Gabor systems with good time-frequency localization properties.

The following result ties tight Gabor frames for $\boldsymbol{L}_2(\mathbb{R})$ to orthonormal Gabor bases for subspaces of $\boldsymbol{L}_2(\mathbb{R})$. It is a simple consequence of Theorem 2.1.

Corollary 4.1 ([28]) *Assume $\|g\| = 1$. $(g, 1/b, 1/a)$ is an orthonormal basis for its closed linear span if and only if (g, a, b) is a tight frame for $\boldsymbol{L}_2(\mathbb{R})$.*

Now, let $\{g_{na,mb}\}$ be a frame for $\boldsymbol{L}_2(\mathbb{R})$ with frame operator S, then a standard way to construct a tight frame is the following. Compute

$$\tilde{g} = S^{-\frac{1}{2}}g \tag{40}$$

then the set $\{\tilde{g}_{na,mb}\}$ is a tight frame for $\boldsymbol{L}_2(\mathbb{R})$ and by Corollary 4.1 (after normalizing $\tilde{g}$) $\{\tilde{g}_{k/b,l/a}\}$ is an orthonormal basis for its closed linear span (equivalently we can apply this orthogonalization procedure directly to the set $\{g_{k/b,l/a}\}$). Our OFDM system is now given by setting $\psi := \tilde{g}$, $T = 1/b$, $F = 1/a$, or equivalently,

$$\psi_{kl} := \tilde{g}_{k/b,l/a}. \tag{41}$$

In order to obtain an OFDM system that is well localized in the time-frequency plane, it seems natural to start with a function g with good time-frequency localization such that (g, a, b) generates a frame for $\boldsymbol{L}_2(\mathbb{R})$, apply the orthogonalization procedure (40) and hope that $\tilde{g} = S^{-\frac{1}{2}}g$ and whence ψ inherit these localization properties. This approach is not new, it has already been considered in [25, 1], however with very different conclusions.

It is stated in [25] that "*such an orthogonalization of the pulses, however, is not desirable ... because the good time-frequency localization of the pulses is destroyed by such a transformation*". In contrast in [1] it is claimed that the orthogonalization procedure (40) "*in practice ... starting from a well-localized initial filter ... yields well-localized orthogonal filters*".

Who is right and who is wrong? The answer is:

Both and nobody – it depends!

The first step to this answer is the following result.

Corollary 4.2 *(i) Let $g, \hat{g} \in \boldsymbol{L}_{1,w}$ and let (g, a, b) generate a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$. Set $\tilde{g} = S^{-\frac{1}{2}}g$. If the weight function w satisfies the GRS-condition, then $\tilde{g}, \hat{\tilde{g}} \in \boldsymbol{L}_{1,w}$.*
(ii) Let $g, \hat{g} \in \boldsymbol{\ell}_{1,w}$ and let (g, a, b) generate a Gabor frame for $\boldsymbol{\ell}_2(\mathbb{Z})$. Set $\gamma = S^{-1}g$ and $\tilde{g} = S^{-\frac{1}{2}}g$. If the weight function w satisfies the GRS-condition, then $\gamma, \hat{\gamma} \in \boldsymbol{\ell}_{1,w}$ and $\tilde{g}, \hat{\tilde{g}} \in \boldsymbol{\ell}_{1,w}$.

Proof: (i) The proof of this result is similar to the proof of Theorem 3.4. We only indicate the necessary modifications. Let Φ be the matrix-valued defining function of a hermitian positive definite Laurent operator L and denote $L^{-\frac{1}{2}} = [C_k]_{k,l=-\infty}^{\infty}$. It follows from the basic properties of block Laurent operators that $\Phi(\omega) > 0$ for all ω and

$$C_k = \int_0^1 \frac{1}{\sqrt{\Phi(\omega)}} \, e^{-2\pi i k \omega} \, d\omega, \qquad k \in \mathbb{Z}.$$

The fact that L is self-adjoint positive-definite implies that $L^{\frac{1}{2}}$ is in the same algebra as L, see [16]. Thus if $L \in \mathcal{W}_w^m$ (and consequently $L^{\frac{1}{2}} \in \mathcal{W}_w^m$) for weights satisfying (30), then $L^{-\frac{1}{2}} \in \mathcal{W}_w^m$. The rest follows now by repeating the steps in Theorem 3.4 and using the remark following Theorem 3.4.
(ii) In this case the frame operator S is a block Laurent operator (see [29]). The assumption $g \in \boldsymbol{\ell}_{1,w}$ implies that $S \in \mathcal{W}_w^m$. Along the same lines as above and in Theorem 3.4 it follows that S^{-1} and $S^{-\frac{1}{2}} \in \mathcal{W}_w^m$ and consequently $\gamma, \hat{\gamma} \in \boldsymbol{\ell}_{1,w}$, similar for $\tilde{g}, \hat{\tilde{g}}$. ◻

Thus if the window (i.e. pulse shape) g satisfies the properties of Corollary 4.2 the orthogonalization procedure (40) yields a function that is in the same

space (in the same *algebra*) as the function g, and the quality of decay does not change.[2]

For a window g with exponential decay (in time and frequency) it has been shown that $\tilde{g}$ also has exponential decay (see [29] for a proof for $\ell_2(\mathbb{Z})$, this result has later been extended to $\boldsymbol{L}_2(\mathbb{R})$, cf. [3]). However, similar to the dual window, in general the exponent for the decay of $\tilde{g}$ will be smaller than for g.

Since the Gaussian is optimally localized in the time-frequency plane, in the sense that it minimizes the uncertainty principle, it is interesting to note that for $g(t) = e^{-\pi t^2}$ the function $\tilde{g}$ has only exponential decay in time and frequency, cf. [3]. Thus we certainly loose some time-frequency localization in this case, although the resulting function $\tilde{g}$ still has exponential decay.

Thus it seems there exists a large class of windows whose time-frequency localization properties are not affected by applying (40). This is however only half of the truth, since in the considerations above we have ignored any constants that come into play. This is acceptable from an asymptotic-analysis viewpoint, but not for applications.

Consider for example the following situation. Take a window g with exponential decay in time and frequency and assume that (g, T, F) constitutes a frame for $TF > 1$. Then for any $\varepsilon > 0$ there exists an N such that for $T > N, F > N$

$$\left\| \frac{g}{\|g\|} - \frac{\tilde{g}}{\|\tilde{g}\|} \right\| \leq \varepsilon$$

since $(TF)\gamma \to g$ for $T \to \infty, F \to \infty$ (see [15] for a mathematically precise formulation). But of course $TF \gg 1$ leads to an unacceptable large loss of capacity for OFDM, already $TF > 2$ seems to be prohibitive in this context.

On the other hand, if we let $TF \to 1$, then $\tilde{g}$ and γ will become increasingly "ill-localized" in the time-frequency plane (since in the limit case $TF = 1$ we are confronted with the Balian-Low theorem), although from theory we know that $\tilde{g}$ has exponential decay as long as $TF > 1$. However, as pointed out earlier, this decay involves a constant C that depends on the ratio of the frame bounds. Since $A \to 0$ for $TF \to 1$ it follows that $C \to \infty$. Thus for applications such as OFDM the exponential-decay property of γ and $\tilde{g}$ quickly becomes meaningless for TF close to 1. If we choose a well-

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[2]Due to the connection between Gabor frames and modulated filter banks [7], Corollary 4.2 also yields a characterization of the decay properties of paraunitary modulated filter banks.

localized window, there is a trade-off between increasing TF and increasing the frame bound ratio B/A.

Hence a correct formulation from a practical viewpoint of the two contradicting statements above is:

> *If g is well localized in time and frequency and if the frame bounds satisfy $B/A \approx 1$, then $\tilde{g}$ is also well localized in time and frequency.*

Numerical experiments indicate that for $TF = 1.3$ or $TF = 1.4$ it is possible to construct OFDM basis functions with good time-frequency localization, e.g. choose the Gaussian as initial window g. A definite answer if the time-frequency properties of the resulting OFDM basis functions obtained in that way are sufficient for practical purposes is difficult, since it depends on the actual ISI and ICI, as well as the AWGN behavior of the channel.

Remark: It is well-known that Wilson bases can be constructed from twofold oversampled tight Gabor frames [10]. In light of this fact Corollary 4.2 provides an extension of existing results on the connection between the decay behavior of Gabor frames and Wilson bases.

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